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FIXED POINT THEORY IN WEAK SECOND-ORDER ARITHMETIC

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We develop a basic part of fixed point theory in the context of weak subsystems of second-order arithmetic. RCA_0 is the system of recursive comprehension and Σ_1^0 induction. WKL_0 is RCA_0 plus the weak König's lemma: *every infinite tree of sequences of* 0's *and* 1's *has an infinite path*. A topological space X is said to possess the *fixed point property* if every continuous function $f: X \to X$ has a point $x \in X$ such that $f(x) = x$. Within WKL_0 (indeed RCA_0), we prove Brouwer's theorem asserting that every nonempty compact convex closed set C in $\mathbb{R}^n$ has the fixed point property, provided that C is expressed as the completion of a countable subset of $\mathbb{Q}^n$. We then extend Brouwer's theorem to its infinite dimensional analogue (the Tychonoff–Schauder theorem for $\mathbb{R}^{\mathbb{N}}$) still within RCA_0. As an application of this theorem, we prove the Cauchy–Peano theorem for ordinary differential equations within WKL_0, which was first shown by Simpson without reference to the fixed point theorem. Within RCA_0, we also prove the Markov–Kakutani theorem which asserts the existence of a common fixed point for certain families of affine mappings. Adapting Kakutani's ingenious proof for deducing the Hahn–Banach theorem from the Markov–Kakutani theorem, we also establish the Hahn–Banach theorem for separable Banach spaces within WKL_0, which was first shown by Brown and Simpson in a different way.

1. Introduction

The purpose of this paper is to develop part of functional analysis concerned with fixed point theorems within a relatively weak subsystem of second-order arithmetic, known as WKL_0. The interest of WKL_0 has been well established through ongoing program, called Reverse Mathematics, whose ultimate goal is to answer the following question: *What set existence axioms are needed to prove the theorems of ordinary mathematics*? For information on the program, see [6], [7], [15], [16].

We here briefly describe the system WKL_0 and two other related systems RCA_0, ACA_0. RCA_0 is the system of recursive comprehension and Σ_1^0 induction. This is the weakest system we shall consider, but is still strong enough to develop some basic theory of continuous functions and countable algebras. The system WKL_0 consists of RCA_0 plus the weak König's lemma: *every infinite tree of sequences of* 0's *and* 1's *has an infinite path*. The first-order part of WKL_0 is the same as that of RCA_0, but WKL_0 proves many important theorems which are not provable in RCA_0, e.g., the Heine–Borel theorem. From the viewpoint of the traditional proof theory, both RCA_0 and WKL_0 are as weak as Primitive Recursive Arithmetic, which is regarded as a formal system suited for Hilbert's finitism to a great extent. The third system ACA_0 consists of RCA_0 plus the arithmetical comprehension axiom. This system is strictly stronger than WKL_0, and its first-order part is just Peano arithmetic.

In this paper, we discuss several forms of fixed point theorems and their applications within WKL_0. A topological space X is said to possess the *fixed point property* if every continuous function $f: X \to X$ has a point $x \in X$ such that $f(x) = x$. An elementary argument in RCA_0 shows that the unit interval $[0, 1]$ has the fixed point property. However, it is not provable within RCA_0 that $[0, 1]^2$ (or the closed unit disc) has the fixed point property. We indeed show that this statement is equivalent to WKL_0 over RCA_0. A general assertion of Brouwer's theorem is that every nonempty compact convex subset C of $\mathbb{R}^n$ has the fixed point property. We prove this assertion within WKL_0, provided that the set C can be expressed as the completion of a countable subset of $\mathbb{Q}^n$ as well as that the complement of C is expressed as the union of (a sequence of) basic open sets. In fact, the assertion with the same proviso can be proved in RCA_0, since if the compact set C contains two or more points, the line segment connecting two distinct points in C must be compact, which implies WKL_0, and otherwise the assertion is trivial. We then extend Brouwer's theorem to its infinite dimensional analogue (the Tychonoff–Schauder theorem for $\mathbb{R}^{\mathbb{N}}$) by adapting Ky Fan's technique based on the Knaster–Kuratowski–Mazurkiewicz theorem (see [5]). As an application of the fixed point theorem for $\mathbb{R}^{\mathbb{N}}$, we prove the Cauchy–Peano theorem for ordinary differential equations within WKL_0, which was first shown by Simpson [14] without reference to the fixed point theorem.

We next discuss the Markov–Kakutani fixed point theorem which asserts the existence of a common fixed point for certain families of affine mappings. While the original proof due to Markov depended on Tychonoff's theorem, Kakutani [11] gave a direct proof. We adapt Kakutani's proof for RCA_0. Kakutani [11] also proved that the Markov–Kakutani theorem implies the Hahn–Banach theorem. We use his technique to reprove the Hahn–Banach theorem for separable Banach spaces within WKL_0, which was first shown in a direct but somewhat unnatural way by Brown and Simpson [3].

In Section 2, we define the formal systems RCA_0, WKL_0 and ACA_0. Section 3 is devoted to the development of basic concepts of real analysis. In Section 4, we discuss uniform continuity and integration. In Section 5, we investigate what set existence axioms are needed to prove some variants of Brouwer's fixed point theorems. In Section 6, we prove, within WKL_0, the Tychnoff–Schauder theorem for $\mathbb{R}^{\mathbb{N}}$, and apply it to the Cauchy–Peano theorem. In Section 7, we prove, within WKL_0, the Markov–Kakutani fixed point theorem for $\mathbb{R}^{\mathbb{N}}$, and then use it to prove the Hahn–Banach theorem for separable Banach spaces within WKL_0.

2. The systems RCA_0, WKL_0 and ACA_0

In this section, we describe the formal systems RCA_0, WKL_0, and ACA_0, following [3], [16]. The reader who is familiar with these systems may skip to the next section.

The language of second-order arithmetic is a two-sorted language with number variables $i, j, k, l, m, \ldots$ and set variables $X, Y, Z, \ldots$. Numerical terms are built up from number variables and constant symbols 0 and 1 by means of binary operations $+$ and $\cdot$. Atomic formulas are $t_1 = t_2$, $t_1 < t_2$, and $t_1 \in X$, where t_1 and t_2 are numerical terms. Formulas are built up from atomic formulas by means of propositional connectives, number quantifiers $\forall n$ and $\exists n$, and set quantifiers $\forall X$ and $\exists X$.

A formula is said to be *arithmetical* if it contains no set quantifiers. An arithmetical formula is said to be Σ_0^0 if all of its number quantifiers are *bounded,* i.e., of the form $\forall n\,(n < t \rightarrow \cdots)$ or $\exists n\,(n < t\ \&\ \cdots)$. An arithmetical formula is said to be Σ_1^0 (resp. Π_1^0) if it is of the form $\exists m\, \phi(m)$ (resp. $\forall m\, \phi(m)$) where $\phi(m)$ is a Σ_0^0 formula.

The system RCA_0 consists of the ordered semiring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ together with the scheme of Δ_1^0 comprehension and Σ_1^0 induction. The Δ_1^0 (*recursive*) *comprehension scheme* consists of all formulas of the form

$$\forall n\,(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X\, \forall n\,(n \in X \leftrightarrow \phi(n)),$$

where $\phi(n)$ is a Σ_1^0 formula, $\psi(n)$ is a Π_1^0 formula, and X does not occur freely in $\phi(n)$. The Σ_1^0 *induction scheme* consists of all formulas of the form

$$\phi(0) \wedge \forall n\,(\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall n\, \phi(n),$$

where $\phi(n)$ is a Σ_1^0 formula. At all times, we assume the law of the excluded middle.

The system ACA_0 consists of RCA_0 plus the *arithmetical comprehension scheme*

$$\exists X\, \forall n\,(n \in X \leftrightarrow \phi(n)),$$

where $\phi(n)$ is arithmetical and X does not occur freely in $\phi(n)$. We can easily see that any arithmetical instance of induction scheme is provable in ACA_0, and that ACA_0 is a conservative extension of first-order Peano arithmetic.

The system WKL_0 is intermediate between RCA_0 and ACA_0. Within RCA_0, we define Seq_2 to be the set of (codes for) finite sequences of 0's and 1's. A set $T \subseteq Seq_2$ is said to be a *tree* if any initial segment of a sequence in T is also in T. A *path* through T is a tree $P \subseteq T$ such that for any two sequences in P, one of them is an initial segment of the other. The axioms of WKL_0 are those of RCA_0 plus weak König's lemma: *every infinite tree* $\subseteq Seq_2$ *has an infinite path.* WKL_0 is known to be conservative over Primitive Recursive Arithmetic with respect to Π_2^0 sentences.

3. The basic concepts of real analysis

This section is devoted to the development of basic concepts of real analysis, including continuous functions, compactness and convexity in the spaces $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$, all within the system RCA_0.

First of all, we use the symbol $\mathbb{N}$ informally to denote the set of natural numbers. We introduce total functions from $\mathbb{N}$ into $\mathbb{N}$ by encoding them as sets of ordered pairs. Within RCA_0, we can define most of elementary numerical functions (e.g., the exponential function m^n) in the usual way, and can prove their basic properties. We then define (codes for) *rational numbers* to be certain ordered pairs of natural numbers. The arithmetical operations on the rational numbers are defined in the standard way. We write $\mathbb{Q}$ for the set (or the field) of rational numbers thus defined.

We define an *(infinite) sequence of rational numbers* to be a function $f:\mathbb{N}\to\mathbb{Q}$, and denote such a sequence by $\langle a_n : n\in\mathbb{N}\rangle$ or simply by $\langle a_n\rangle$, where $a_n = f(n)$. A *real number* is defined to be a sequence $\langle a_n\rangle$ of rational numbers such that $\forall n\,\forall i\,(|a_n - a_{n+i}| \leq 2^{-n})$. We use $\mathbb{R}$ informally to denote the set of all real numbers. Note that $\mathbb{R}$ does not formally exist as a set within RCA_0. Two real numbers $\langle a_n\rangle$ and $\langle b_n\rangle$ are defined to be equal, $\langle a_n\rangle = \langle b_n\rangle$, iff $\forall n\,(|a_n - b_n| \leq 2^{-n+1})$. The relation $<$ is defined by $\langle a_n\rangle < \langle b_n\rangle$ iff $\exists n\,(b_n - a_n > 2^{-n+1})$. Then it is easy to see that for any two real numbers $\langle a_n\rangle$ and $\langle b_n\rangle$, exactly one of the following holds (in RCA_0): $\langle a_n\rangle < \langle b_n\rangle$, $\langle a_n\rangle = \langle b_n\rangle$, $\langle a_n\rangle > \langle b_n\rangle$. We let $0 = \langle 0 : n\in\mathbb{N}\rangle$ and $1 = \langle 1 : n\in\mathbb{N}\rangle$. The operations $+$ and $\cdot$ are defined by

$$\langle a_n\rangle + \langle b_n\rangle = \langle a_{n+1} + b_{n+1}\rangle,$$

$$\langle a_n\rangle \cdot \langle b_n\rangle = \langle a_{n+m} \cdot b_{n+m}\rangle,$$

where m is the least number such that $\max(|a_0|, |b_0|) + 1 \leq 2^{m-1}$. Within RCA_0, one can prove that $(\mathbb{R}, +, \cdot, 0, 1, <)$ is an Archimedean ordered field.

An *infinite sequence of real numbers* is defined to be a doubly indexed sequence of rational numbers $\langle a_{mn} : m, n\in\mathbb{N}\rangle$ such that for each m, $\langle a_{mn} : n\in\mathbb{N}\rangle$ is a real number. Such a sequence of real numbers is also denoted $\langle x_m : m\in\mathbb{N}\rangle$, where $x_m = \langle a_{mn} : n\in\mathbb{N}\rangle$. We write $\mathbb{R}^{\mathbb{N}}$ for the set of infinite sequences of real numbers. Similarly, an *n-tuple* (or *finite sequence with length n*) *of real numbers*, for $n \geq 1$, is a doubly indexed sequence of rational numbers $\langle a_{ij} : i < n, j\in\mathbb{N}\rangle$ such that for each $i < n$, $\langle a_{ij} : j\in\mathbb{N}\rangle$ is a real number. We write $\mathbb{R}^n$ $(n \geq 1)$ for the set of n-tuples of real numbers. In case $n = 1$, $\mathbb{R}^n$ is identified with $\mathbb{R}$. Two elements in $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$) are defined to be equal, if their corresponding components are equal with respect to the equality of $\mathbb{R}$.

We define $\max : \mathbb{R}^n \to \mathbb{R}$ and $\min : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\max(\langle\langle a_{ij} : j\in\mathbb{N}\rangle : i < n\rangle) = \langle \max\{a_{ij} : i < n\} : j\in\mathbb{N}\rangle,$$

$$\min(\langle\langle a_{ij} : j\in\mathbb{N}\rangle : i < n\rangle) = \langle \min\{a_{ij} : i < n\} : j\in\mathbb{N}\rangle.$$

It is easy to check that the sequences on the right sides are 'real numbers'. The finite sum $\sum : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\sum (\langle \langle a_{ij} : j \in \mathbb{N} \rangle : i < n \rangle) = \left\langle \sum_{i<n} a_{i(j+n-1)} : j \in \mathbb{N} \right\rangle.$$

Its well definedness is also clear. Although max, min and $\sum$ could be defined as continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ (the notion of continuous functions will be given later), we treat them like operations such as $+$ and $\cdot$ on $\mathbb{R}$.

The vector addition and scalar multiplication on $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$ are defined in an obvious way. We define the norm $\|\cdot\|_n$ on $\mathbb{R}^n$ by

$$\| \langle x_i : i < n \rangle \|_n = \max_{i<n} |x_i|.$$

Thus $\mathbb{R}^n$ can be viewed as a separable Banach space. We also use $\|\cdot\|_n$ as a seminorm on $\mathbb{R}^{\mathbb{N}}$ by letting $\|\langle x_i : i \in \mathbb{N} \rangle\|_n = \|\langle x_i : i < n \rangle\|_n$. Then $\mathbb{R}^{\mathbb{N}}$ is a linear space with countably many seminorms, and indeed a separable Fréchet space.

We next discuss the topology on $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$. Let $\mathbb{Q}^n$ ($n \geq 1$) be the set of (codes for) finite sequences of rational numbers with length n. $\mathbb{Q}^n$ may be regarded as a subset of $\mathbb{R}^n$. We assume that $\mathbb{Q}^n$ and $\mathbb{Q}^m$ are disjoint if $n \neq m$. We then put $\mathbb{Q}^{<\mathbb{N}} = \bigcup_{n \geq 1} \mathbb{Q}^n$. A code for a *basic open set* $B_r(a)$ in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is an ordered pair (a, r) with $a \in \mathbb{Q}^n$ (resp. $\mathbb{Q}^{<\mathbb{N}}$) and $r \in \{0\} \cup \mathbb{Q}^+$ (the positive rationals). For $a \in \mathbb{Q}^{<\mathbb{N}}$, we define $\dim(a)$ to be the dimension of a, i.e., $a \in \mathbb{Q}^{\dim(a)}$. A point $x \in \mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is said to *belong to* a basic open set $B_r(a)$ in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$), denoted $x \in B_r(a)$, if $\|x - a\|_n < r$ (resp. $\|x - a\|_{\dim(a)} < r$). By $x \in \overline{B_r(a)}$, we mean $\|x - a\|_{\dim(a)} \leq r$. We write $B_r(a) \subseteq B_s(b)$ in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) to mean that $\|b - a\|_n + r \leq s$ (resp. $\dim(b) \leq \dim(a)$ and $\|b - a\|_{\dim(b)} + r \leq s$). A basic open set $B_r(a)$ is said to be *nonempty* if $r > 0$.

A code for an *open set* U in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is a sequence of (codes for) basic open sets in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$), i.e., a function $\phi : \mathbb{N} \to \mathbb{Q}^n \times \mathbb{Q}$ (resp. $\phi : \mathbb{N} \to \mathbb{Q}^{<\mathbb{N}} \times \mathbb{Q}$) such that for each $n \in \mathbb{N}$, $\phi(n)$ is a code for a basic open set in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$). A point $x \in \mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is said to *belong to* an open set U with code ϕ in $\mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$), denoted $x \in U$, if it belongs to a basic open set in the sequence, i.e., there exists $n \in \mathbb{N}$ such that $x \in \phi(n)$. This definition of open sets is not identical with the corresponding definition in [3], but the equivalence of the two definitions can be easily established within RCA_0.

We define *closed sets* to be just complements of open sets. Remark that a closed set has only negative information on its members, and so, in general, it is difficult to deal with points in a closed set. We thus introduce the notion of countably representable closed sets, which have both positive and negative information on their members. Let f be a function from $\mathbb{N}$ to $\mathbb{R}^n$ (or $\mathbb{R}^{\mathbb{N}}$). We informally identify f with its range, say A, although A may not exist as a set. A closed set $C \subseteq \mathbb{R}^n$ (resp. $C \subseteq \mathbb{R}^{\mathbb{N}}$) is said to be *countably represented* by $A \subseteq \mathbb{R}^n$

(resp. $A \subseteq \mathbb{R}^{\mathbb{N}}$) denoted $C = \hat{A}$, if for all $x \in \mathbb{R}^n$ (resp. $x \in \mathbb{R}^{\mathbb{N}}$),

$$x \in C \iff \text{there exists an infinite sequence } \langle a_i \rangle \text{ from } A \text{ such that for each } i,\ \|x - a_i\|_n \leq 2^{-i} \text{ (resp. } \|x - a_i\|_{i+1} \leq 2^{-i}).$$

For example, $[0, 1]^n$ is countably represented by $A_n = \{\langle q_0, \ldots, q_{n-1}\rangle \in \mathbb{Q}^n : 0 \leq q_i \leq 1 \text{ for each } i < n\}$, and $[0, 1]^{\mathbb{N}}$ by $\{\langle q_0, q_1, \ldots, q_n, 0, 0, \ldots\rangle \in \mathbb{Q}^{\mathbb{N}} : 0 \leq q_i \leq 1 \text{ for each } i \leq n\}$.

A code for a *continuous partial function* $f : \mathbb{R}^n \to \mathbb{R}^m$ (or $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$) is a sequence Φ of pairs of nonempty basic open sets such that

(i) $(B_r(a), B_{s_1}(b_1)) \in \Phi \;\&\; (B_r(a), B_{s_2}(b_2)) \in \Phi \to \|b_1 - b_2\|_k < s_1 + s_2$, where $k = \min(\dim(b_1), \dim(b_2))$,

(ii) $(B_r(a), B_{s_1}(b_1)) \in \Phi \;\&\; B_{s_1}(b_1) \subseteq B_{s_2}(b_2) \to (B_r(a), B_{s_2}(b_2)) \in \Phi$,

(iii) $(B_{r_1}(a_1), B_s(b)) \in \Phi \;\&\; B_{r_2}(a_2) \subseteq B_{r_1}(a_1) \to (B_{r_2}(a_2), B_s(b)) \in \Phi$.

Although Φ is formally a function from $\mathbb{N}$ to $\mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q}^m \times \mathbb{Q}^+$ (resp. $\mathbb{Q}^{<\mathbb{N}} \times \mathbb{Q}^+ \times \mathbb{Q}^{<\mathbb{N}} \times \mathbb{Q}^+$), we write $(B_r(a), B_s(b)) \in \Phi$ if $\Phi(n) = (a, r, b, s)$ for some $n \in \mathbb{N}$. Intuitively, $(B_r(a), B_s(b)) \in \Phi$ means that $f(B_r(a)) \subseteq \overline{B_s(b)}$ where f is the continuous partial function encoded by Φ. A point $x \in \mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is said to *belong to* the *domain* of function f with code Φ if for all $\epsilon > 0$ (resp. for all $\epsilon > 0$ and all $i \in \mathbb{N}$), there exists $(B_r(a), B_s(b)) \in \Phi$ such that $x \in B_r(a)$ and $s < \epsilon$ (resp. $s < \epsilon$ and $\dim(b) \geq i + 1$). If a point $x \in \mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) is in the domain of f, we define $f(x)$ to be the point $y \in \mathbb{R}^n$ (resp. $\mathbb{R}^{\mathbb{N}}$) such that if $x \in B_r(a) \;\&\; (B_r(a), B_s(b)) \in \Phi$ then $y \in \overline{B_s(b)}$. We can prove, within RCA_0, that such a y exists uniquely (up to the equality of points). A continuous partial function $f : \mathbb{R}^n \to \mathbb{R}^m$ or $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is said to be a *continuous function from* a closed set C *to* a closed set D, if the domain of f includes C and for each $x \in C$, $f(x) \in D$.

In the rest of this section, we discuss the notions of compactness and convexity, which play the most important roles in fixed point theory. Let C be a closed set in $\mathbb{R}^n$ or $\mathbb{R}^{\mathbb{N}}$. C is said to be *compact* (in the sense of Heine–Borel) iff for any open set $\langle B_i : i \in \mathbb{N}\rangle$ with B_i basic open, if $\langle B_i : i \in \mathbb{N}\rangle$ covers C (i.e., all points in C belong to some B_i) then there exists $n \in \mathbb{N}$ such that $\langle B_i : i < n\rangle$ covers C. The notion of compactness should be distinguished from related concepts such as the Bolzano–Wierstrass property: *every bounded sequence has a limit point.* H. Friedman [7] has shown that WKL_0 and the compactness (in the sense of Heine–Borel) of $[0, 1]$ are equivalent over RCA_0, and that ACA_0 and the Bolzano–Wierstrass property of $[0, 1]$ are equivalent over RCA_0.

We here state the following lemma without proof.

3.1. Lemma (RCA_0). *The following are pairwise equivalent*:

(i) WKL_0,

(ii)$_n$ $[0, 1]^n \subseteq \mathbb{R}^n$ *is compact*, $n \geq 1$,

(iii) $[0, 1]^{\mathbb{N}}$ *is compact.*

For the proof, see Lemma 2.4 and 3.3 of Simpson [14] and the references given there.

We finally discuss the notion of convexity. Let C be a closed set in $\mathbb{R}^n$ or $\mathbb{R}^{\mathbb{N}}$. We define C to be *convex* iff for all x, y in C and for all $q \in [0, 1] \cap \mathbb{Q}$, $qx + (1-q)y \in C$. Then the following holds.

3.2. Lemma (RCA$_0$). *Let C be a convex closed set in $\mathbb{R}^n$ or $\mathbb{R}^{\mathbb{N}}$. Then for any finite subset $\{x_0, \dots, x_{n-1}\} \subseteq C$ and for any set of non-negative reals $\{\alpha_0, \dots, \alpha_{n-1}\}$ with $\sum_{i<n} \alpha_i = 1$, we have*

$$\sum_{i<n} \alpha_i x_i \in C.$$

Proof. Fix any $\{x_0, \dots, x_{n-1}\} \subseteq C$. We first prove the following statement by induction on $k \leqslant n$:

(∗) for any non-negative rationals $\{q_0, \dots, q_{k-1}\}$ with $\sum_{i<k} q_i = 1$,

$$\sum_{i<k} q_i x_i \in C.$$

We here notice that the statement (∗) is Π_1^0, and so we can use the Π_1^0 induction (which is equivalent to Σ_1^0 induction, see [8, Lemma 1.1]). Assume (∗) holds for k. Let $\{q_0, \dots, q_k\}$ be a set of non-negative rationals such that $\sum_{i\leqslant k} q_i = 1$. We may assume $q_k \neq 1$. For if $q_k = 1$ then $q_i = 0$ for $i < k$, and so $\sum_{i\leqslant k} q_i x_i = x_k \in C$. Now consider the set of rationals

$$\left\{\frac{q_0}{1-q_k}, \frac{q_1}{1-q_k}, \dots, \frac{q_{k-1}}{1-q_k}\right\}.$$

Then we have

$$\frac{q_0}{1-q_k} + \frac{q_1}{1-q_k} + \dots + \frac{q_{k-1}}{1-q_k} = \frac{\sum_{i\leqslant k} q_i - q_k}{1-q_k} = 1.$$

So by the induction hypothesis,

$$\sum_{i<k} \frac{q_i}{1-q_k} x_i \in C.$$

Finally, by the convexity of C, we have

$$\sum_{i\leqslant k} q_k x_k = (1-q_k)\left(\sum_{i<k} \frac{q_i}{1-q_k} x_i\right) + q_k x_k \in C.$$

Thus, for any non-negative rationals $\{q_0, \dots, q_{n-1}\}$ with $\sum_{i<n} q_i = 1$,

$$\sum_{i<n} q_i x_i \in C.$$

Since C is closed, we can easily show that for any non-negative reals $\{\alpha_0, \dots, \alpha_{n-1}\}$ with $\sum_{i<n} \alpha_i = 1$,

$$\sum_{i<n} \alpha_i x_i \in C.$$

This completes the proof. □

4. Uniform continuity and integration

In this section, we discuss some important properties of uniformly continuous functions, and then define the Riemann integration for those functions.

We begin with the following definition (cf. Aberth [1], Simpson [14]). Let f be a continuous function from a closed set C ($\subseteq \mathbb{R}^n$) to $\mathbb{R}^m$. Then f is said to be *weakly uniformly continuous* on C if for any $e \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that for all x and y in C, if $\|x-y\|_n < 2^{-d}$ then $\|f(x)-f(y)\|_m < 2^{-e}$. f is said to be *uniformly continuous* on C if there exists a total function $h:\mathbb{N}\to\mathbb{N}$ such that for all $e \in \mathbb{N}$ and all x and y in C, if $\|x-y\|_n < 2^{-h(e)}$ then $\|f(x)-f(y)\|_m < 2^{-e}$. Such a function h is called a *modulus of uniform continuity* for f.

4.1. Lemma (RCA_0). *Let C be a compact closed set in $\mathbb{R}^n$. Then any continuous function $f:C\to\mathbb{R}^m$ is weakly uniformly continuous on C.*

Proof. Let $\Phi = \{(B_i, B_i'): i \in \mathbb{N}\}$ be a code for a continuous function $f:C$ ($\subseteq \mathbb{R}^n$)$\to\mathbb{R}^m$, where B_i and B_i' are basic open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Fix any $\epsilon > 0$. Let

$$\mathcal{B} = \{B : (B, B') \in \Phi \text{ and } B' = (a, r) \text{ with } r < \epsilon/2\}.$$

From the definition of the domain of a continuous function, it is easy to see that $\mathcal{B}$ is an open covering of C. So by the compactness of C, there exists a finite subset of $\mathcal{B}$ which also covers C. Let $\{(a_0, r_0), (a_1, r_1), \ldots, (a_{k-1}, r_{k-1})\}$ be such a finite covering. Now put

$$\mathcal{C} = \{(a_i, r_i - 2^{-l}) : r_i - 2^{-l} > 0,\ i < k,\ l \in \mathbb{N}\}.$$

It can be shown that $\mathcal{C}$ is an open covering of C, too. Again by the compactness of C, there is an $L \in \mathbb{N}$ such that $\mathcal{C}_L = \{(a_i, r_i - 2^{-l}) : r_i - 2^{-l} > 0,\ i < k,\ l \leq L\}$ also covers C. We finally set $\delta = 2^{-L}$, and show that for all x and y in C, if $\|x-y\|_n < \delta$ then $\|f(x)-f(y)\|_m < \epsilon$. Choose any two points x and y from C such that $\|x-y\|_n < \delta$. Since $\mathcal{C}_L$ covers C, there exists a basic open set $(a_i, r_i - 2^{-l})$ in $\mathcal{C}_L$ which the point x belongs to. Then both x and y belong to the basic open set (a_i, r_i) in $\mathcal{B}$, since $\|x-y\|_n < 2^{-L} \leq 2^{-l}$. Hence, by the definition of $\mathcal{B}$, both $f(x)$ and $f(y)$ belong to a basic open set (a, r) with $r < \epsilon/2$, that is, $\|f(x)-f(y)\|_m < \epsilon$. This completes the proof. □

4.2. Lemma (WKL_0). *Let C be an n-dimensional rectangle $\{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ with $a_i, b_i \in \mathbb{R}$. Then any continuous function $f:C\to\mathbb{R}^m$ is uniformly continuous on C.*

Proof. Let f be a continuous function from the rectangle C to $\mathbb{R}^m$. Within WKL_0, a rectangle C is compact by Lemma 3.1. So we know from Lemma 4.1, that for each e, there exists d such that

$$\|x-y\|_n < 2^{-d} \quad\Rightarrow\quad \|f(x)-f(y)\|_m < 2^{-e}.$$

Look back at the proof of Lemma 4.1. In the proof, the compactness of C is used twice, first for the covering $\mathscr{B}$ and second for $\mathscr{C}$. If C is a rectangle $[a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$ with a_i, $b_i \in \mathbb{Q}$, we can easily decide whether a given finite subset of $\mathscr{B}$ (and $\mathscr{C}$) covers C or not, and thus obtained d ($= L$ in the proof of Lemma 4.1) from e in a recursive way. We now want to reduce the general case (a_i, $b_i \in \mathbb{R}$) to this special case.

Suppose for each $i < n$, $a_i = \langle a_{ij} : j \in \mathbb{N} \rangle$, $b_i = \langle b_{ij} : j \in \mathbb{N} \rangle$ with a_{ij}, $b_{ij} \in \mathbb{Q}$. For each $j \in \mathbb{N}$, let C_j be the rectangle

$$\{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : a_{ij} - 2^{-j} \leq x_i \leq b_{ij} + 2^{-j}\}.$$

Then it is easy to see that $C = \bigcap_{j \in \mathbb{N}} C_j$. Define $\mathscr{B}$ as in the proof of Lemma 4.1. Since $\mathscr{B}$ is an open covering of C and C is compact, there exists $j_0 \in \mathbb{N}$ such that $\mathscr{B}$ covers $\bigcap_{j<j_0} C_j$. So we can effectively find a finite subset $\{(a_0, r_0), (a_1, r_1), \ldots, (a_{k-1}, r_{k-1})\} \subseteq \mathscr{B}$ which covers $\bigcap_{j<k} C_j$. Define $\mathscr{C}$ as before. Now $\mathscr{C}$ is an open covering of $\bigcap_{j<k} C_j$. We can then find a finite subcovering $\mathscr{C}_L$ in an effective way. Thus, in WKL_0, there exists a total function $h : \mathbb{N} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$ and for all x, $y \in C$,

$$\|x - y\|_n < 2^{-h(e)} \quad \Rightarrow \quad \|f(x) - f(y)\|_m < 2^{-e}. \qquad \square$$

For the reversal of the above lemma, see Aberth [1, Theorem 7.3] and Simpson [13]. They indeed show that WKL_0 and the statement that any continuous function on $[0, 1]$ is uniformly continuous are equivalent over RCA_0.

The next lemma shows that a uniformly continuous function from a countably represented closed set $\hat{A}$ ($\subseteq \mathbb{R}^n$) to $\mathbb{R}^m$ can be uniquely encoded by its restriction to A. Since a function from A to $\mathbb{R}^m$ is just a point in $(\mathbb{R}^m)^{\mathbb{N}} \approx \mathbb{R}^{\mathbb{N}}$, this lemma is very useful to deal with certain function spaces (cf. the discussion before Theorem 6.2).

4.3 Lemma (RCA_0). *Let C be a closed set in $\mathbb{R}^n$. Suppose that C is countably represented by $A \subseteq \mathbb{Q}^n$. Let $f : A \to \mathbb{R}^m$ be a uniformly continuous function with a modulus function h (i.e., for all a and b in A, $\|a - b\|_n < 2^{-h(e)} \Rightarrow \|f(a) - f(b)\|_m < 2^{-e}$). Then there exists (a code for) a unique continuous function $\hat{f} : \hat{A} \to \mathbb{R}^m$ such that $\hat{f}(a) = f(a)$ for all a in A.*

Proof. Let $f : A \to \mathbb{R}^m$ be a uniformly continuous function with modulus h. Let Φ ($\subseteq \mathbb{Q}^n \times \mathbb{Q}^+ \times \mathbb{Q}^m \times \mathbb{Q}^+$) be a code for a continuous function $\hat{f} : \hat{A} \to \mathbb{R}^m$ defined by

$$(a, r, b, s) \in \Phi \leftrightarrow \exists e \, (r < 2^{-h(e)} \quad \text{and} \quad \|f(a) - b\|_m < s - 2^{-e}).$$

Note that the above definition is Σ_1^0, and so Φ can be seen as a recursive enumeration of its members. It is easy to see that Φ satisfies all the conditions to be a continuous function code. It is also clear that $\hat{f}(a) = f(a)$ for all a in A. $\square$

We next discuss the integration of a uniformly continuous function on a closed interval $[a, b]$ with a, $b \in \mathbb{R}$. For simplicity, we do not deal with multi-variable functions here. Recall that a continuous function on $[a, b]$ is always uniformly continuous in WKL_0, but not always in RCA_0.

Let $f:[a, b]\to R$ be uniformly continuous with a modulus function h. We define a function $S:\mathbb{N}\to\mathbb{R}$ by

$$S(m)=\frac{b-a}{2^m}\sum_{k=0}^{2^m-1} f\left(a+k\frac{k-a}{2^m}\right).$$

Then the *(Riemann) integration* of f on $[a, b]$, denoted $\int_a^b f(x)\,\mathrm{d}x$, is defined to be $\lim_{n\to\infty} S(h(n))$. To see the existence of such a limit, we first show the following lemma.

4.4. Lemma (RCA_0). *If a sequence* $\langle x_n\rangle \in \mathbb{R}$ *satisfies* $\exists n_0 \forall n \forall i\, (|x_n - x_{n+i}| \leqslant 2^{n_0-n})$, *then there exists* $x\in\mathbb{R}$ *such that* $\forall m \exists M \forall n \geqslant M\, (|x-x_n|\leqslant 2^{-m})$. (Cf. Brown and Simpson [3, Lemma 4.1].)

Proof. For $n\in\mathbb{N}$, let $x_n=\langle q_{n,k}:k\in\mathbb{N}\rangle$ with $q_{n,k}\in\mathbb{Q}$. Let $x=\langle q_{n+n_0+2,n+n_0+2}:n\in\mathbb{N}\rangle$. Then it is easy to see that $x\in\mathbb{R}$ and $\forall n\geqslant m+n_0+1$ $(|x-x_n|\leqslant 2^{-m})$. $\square$

Choose $m_0\in\mathbb{N}$ such that $|b-a|\leqslant 2^{m_0}$. Then by the uniform continuity of f,

$$\text{for any } x\in\left[a+k\frac{b-a}{2^{h(n)+m_0}},\, a+(k+1)\frac{b-a}{2^{h(n)+m_0}}\right),$$

we have

$$\left|f(x)-f\left(a+k\frac{b-a}{2^{h(n)+m_0}}\right)\right|<2^{-n}.$$

Hence, for all $i\in\mathbb{N}$,

$$\begin{aligned}|S(h(n+i)-m_0)-S(h(n)+m_0)| &< \frac{|b-a|}{2^{h(n+i)+m_0}}(2^{h(n+i)+m_0}\cdot 2^{-n})\\ &=|b-a|\cdot 2^{-n}\leqslant 2^{m_0-n}.\end{aligned}$$

Therefore, by the above lemma, $\lim_{n\to\infty} S(h(n))=\lim_{n\to\infty} S(h(n)+m_0)$ exists.

5. Brouwer's fixed point theorem

In this section, we investigate what set existence axioms are needed to prove some variants of Brouwer's fixed point theorem. We begin with the following theorem.

5.1. Brouwer's Theorem I. (a)(RCA_0). *For any continuous function* $f:[0,1]\to[0,1]$, *there is a point* $x\in[0,1]$ *such that* $f(x)=x$.

(b) (WKL_0). *For any continuous function* $f:[0,1]^n\to[0,1]^n$, *there is a point* $x\in[0,1]^n$ *such that* $f(x)=x$ $(n\geqslant 2)$.

Proof. (a) We imitate Simpson's proof for the intermediate value theorem [15]. Suppose that for all rational $q\in[0,1]$, $f(q)\neq q$. With the Δ_1^0 comprehension, we define a nested sequence of rational intervals as follows:

$$[a_0,b_0]=[0,1],$$

$$[a_{n+1},b_{n+1}]=\begin{cases}[(a_n+b_n)/2,\,b_n] & \text{if } f((a_n+b_n)/2)>(a_n+b_n)/2,\\ [a_n,\,(a_n+b_n)/2] & \text{if } f((a_n+b_n)/2)<(a_n+b_n)/2.\end{cases}$$

By nested interval convergence (see Lemma 2.2 in Simpson [14]), there exists a real x such that $x=\lim a_n=\lim b_n$. This x is a fixed point for f.

(b) Among several known proofs of this theorem, one given by D. Gale [9] seems to be most easily carried out within WKL_0. His proof mostly consists of manipulations of finite objects (HEX), which do not use any set existence axiom. The only infinitary argument or fact one needs is that every continuous function on $[0,1]^n$ is uniformly continuous, which is proved in our Lemma 4.2. For details, see Gale [9]. □

We remark that part (b) is not provable over RCA_0. In fact, we have

5.2 Theorem (RCA_0). *The following are pairwise equivalent*:

(i) WKL_0,

(ii) *for any continuous function* $f:[0,1]^2\to[0,1]^2$, *there is a point* $x\in[0,1]^2$ *such that* $f(x)=x$.

Proof. Since (i)→(ii) is already proved by Theorem 5.1(b), we only show (ii)→(i). Our argument is essentially due to V. P. Orekov (cf. Chapter IV of Beeson [2]).

By way of contradiction, deny (i). Then $[0,1]$ is not compact by Lemma 3.1. Let $\langle I_i : i\in\mathbb{N}\rangle$ be a sequence of rational open intervals which covers $[0,1]$ but has no finite subcover. From $\langle I_i\rangle$, we can easily construct a sequence of rational closed intervals $\langle J_i : i\in\mathbb{N}\rangle$ such that $\phi\neq J_i\subseteq[0,1]$, $\langle J_i\rangle$ covers $[0,1]$, and any two distinct J_i's are disjoint or have only an endpoint in common. We may assume that J_0 has 0 as its left endpoint and J_1 has 1 as its right endpoint. Define A_k to be the union of all $J_i\times J_k$ and $J_k\times J_i$ for $i\leqslant k$.

From now, we construct a retraction f of $[0,1]^2$ onto the four side of the square $[0,1]^2$. If such an f is constructed, (ii) does not hold. For if r is the rotation of 90° about the point $(\frac{1}{2},\frac{1}{2})$, $r\circ f$ is a continuous function from $[0,1]^2$ into itself which has no fixed point.

We define a retraction f in stages. Suppose f has been defined on all A_i for $i<k$. We want to define f for A_k, compatibly with the value already assigned. Decompose A_k into its connected components $P_1, \dots, P_l$. We can easily observe that each P_i has at least one free side on which P_i does not adjoin $\bigcup_{i<k} A_i$ or any side of $[0, 1]^2$. Now f can be extended to P_i by combining a retraction of P_i onto the sides on which the values of f are already determined (or onto any one side if no such side). Since $\bigcup_{k\in\mathbb{N}} A_k = [0, 1]^2$, this procedure clearly defines a retraction of $[0, 1]^2$ onto its sides. More formally, we have to construct a code for this retraction. This is done by enumerating the pairs of nonempty basic open sets (B_1, B_2) such that $B_1 \subseteq \bigcup_{i<k} A_i$ for some k and $f(B_1) \subseteq \overline{B_2}$. Since this construction is just a routine, we omit the details. □

We now generalize Theorem 5.1 as follows:

5.3. Brouwer's Theorem II (WKL$_0$). *Let $\{a_0, \dots, a_k\}$ be a finite subset of $\mathbb{Q}^n$. Let C be the closed set $\{\sum_{i\leq k} \alpha_i a_i : \alpha_i \in \mathbb{R},\ \alpha_i \geq 0$ for $i \leq k$, and $\sum_{i\leq k} \alpha_i = 1\}$. Then for any continuous function $f: C \to C$, there is a point $x \in C$ such that $f(x) = x$.*

Proof. As in the standard proof (see [17]), we will show that C is a retract of a sufficient large n-dimensional rectangle (in a certain coordinate system). Changing coordinate systems is not essential, but makes it much easier to construct such a retraction.

We first find a basis for the space L spanned by $\{a_1 - a_0, a_2 - a_0, \dots, a_k - a_0\}$. This can be done by simple calculations of rational matrices (e.g., Gaussian elimination). Notice that the assumption $\{a_0, a_1, \dots, a_k\} \subseteq \mathbb{Q}^n$ (rather than $\subseteq \mathbb{R}^n$) is necessary to determine whether some entries of the matrices in the computations are zero or not.

Let $\mathcal{B}$ be a base for $\mathbb{R}^n$ including the base for the subspace L spanned by $\{a_1 - a_0, a_2 - a_0, \dots, a_k - a_0\}$. Using the coordinate system relative to the base $\mathcal{B}$, the points $a_0, a_1, \dots, a_k$ can be expressed as n-tuples in $\mathbb{R}^l \times \{0\}^{n-l}$, where l is the dimension of the subspace L. So there exists $d \in \mathbb{R}$ such that the convex hull C of $\{a_0, a_1, \dots, a_k\}$ is included in $[-d, d]^l \times \{0\}^{n-l} \subseteq [-d, d]^n$ with respect to the new coordinate system. There is an obvious retraction from $[-d, d]^n$ onto $[-d, d]^l \times \{0\}^{n-l}$. So if we can show that C is a retract of $[-d, d]^l \times \{0\}^{n-l}$, then we may conclude that C has the fixed point property (i.e., any continuous function from C to itself has a fixed point) since we already know from Theorem 5.1 that $[-d, d]^n$ has the fixed point property. Remark that if C has the fixed point property in the new coordinate system then it also has in the standard coordinate system, since a continuous function code in the standard coordinate system can be easily translated in the new coordinate system (and vice versa).

From now on, we regard C as a subset of $[-d, d]^l$ by ignoring the zeros in the $(l+1)$-th to n-th coordinates. To clarify the shape of C, we remove all the superfluous points from $\{a_0, a_1, \dots, a_k\}$ and construct the smallest subset

$S \subseteq \{a_0, a_1, \ldots, a_k\}$ such that the convex hull of S is still C. Note that a_{i_0} is superfluous if there are $l+1$ points $a_{i_1}, a_{i_2}, \ldots, a_{i_{l+1}}$ in $\{a_0, a_1, \ldots, a_k\}$ such that $a_{i_0} \neq a_{i_j}$ for all $j \neq 0$, and such that a_{i_0} is involved in the convex hull of $\{a_{i_1}, a_{i_2}, \ldots, a_{i_{l+1}}\}$.

We may assume that $\{a_0, \ldots, a_k\}$ has no superfluous points. Let $\bar{a}$ be the center of C, i.e., $\bar{a} = (\sum_{i \le k} a_i)/(k+1)$. We construct a retraction $\hat{g}: [-d, d]^l \to C$ as follows. For $b \in [-d, d]^l \cap \mathbb{Q}^l$, if $b \in C$ then we put $g(b) = b$, and if $b \notin C$ then we put $g(b) =$ the point at which the line segment connecting b and $\bar{a}$ intersects a face of C. Note that such an intersection can be obtained by solving a system of linear equations with rational coefficients. The function g thus defined on $[-d, d]^l \cap \mathbb{Q}^l$ can be uniquely extended to a continuous function $\hat{g}$ on $[-d, d]^l$. In fact, a code Φ for the continuous function $\hat{g}$ is defined by

$$\begin{aligned}(B, B') \in \Phi \quad \Leftrightarrow \quad & \exists b_0, b_1, \ldots, b_l \in [-d, d]^l \cap \mathbb{Q}^l \text{ such that} \\ & B \text{ is included in the convex hull of } \{b_0, \ldots, b_l\} \\ & \text{and } \{g(b_0), \ldots, g(b_l)\} \text{ is included in } B'.\end{aligned}$$

Then we can easily see that Φ is indeed a code for the desired retraction. This completes the proof. □

The above theorem can be further generalized to Theorem 5.4 and Theorem 6.1. Since the next theorem can be proved in the same way as Theorem 6.1, we just state it without proof.

5.4. Brouwer's Theorem III (WKL_0). *Let C be a nonempty compact convex closed set in $\mathbb{R}^n$, which is also assumed to be countably represented by $A \subseteq \mathbb{Q}^n$. Then any continuous function $f: C \to C$ has a fixed point.*

6. Tychonoff–Schauder theorem for $\mathbb{R}^{\mathbb{N}}$ and its application to ordinary differential equations

In this section, we extend Brouwer's theorem to its infinite dimensional analogue (Tychonoff–Schauder theorem for $\mathbb{R}^{\mathbb{N}}$), from which we prove the Cauchy–Peano theorem for ordinary differential equations.

6.1. Tychonoff–Schauder Theorem (WKL_0). *Let C be a nonempty compact convex closed set in $\mathbb{R}^{\mathbb{N}}$, which is also assumed to be countably represented by $A \subseteq \mathbb{Q}^{\mathbb{N}}$. Then any continuous function $f: C \to C$ has a fixed point.*

Proof. By way of contradiction, we assume that for all $x \in C$, $f(x) \neq x$. Let Φ be a code for f. Recall that $(B, B') \in \Phi$ means that $f(B) \subseteq \overline{B'}$. Let $\mathscr{B} = \{B$: there is B' such that $(B, B') \in \Phi$ and $B \cap B' = \emptyset\}$. It is easy to see that $\mathscr{B}$ covers C, since

$f(x) \neq x$ for all $x \in C$. By the compactness of C, $\mathscr{B}$ has a finite subcover $\{B_0, B_1, \ldots, B_k\}$. So there exists $n \geqslant 1$ and $\epsilon > 0$ such that for all $x \in C$, $\|f(x) - x\|_n > \epsilon$. We may assume ϵ is a rational number. Fix n and ϵ. For contradiction, we will show that there exists $x \in C$ such that $\|f(x) - x\|_n \leqslant \epsilon$. We define, for each $a \in A$,

$$Ta = \{B\text{: there exists } B' = (b, s) \text{ with } \dim(b) \geqslant n \\ \text{such that } (B, B') \in \Phi \text{ and } \|b - a\|_n + s \leqslant \epsilon\}.$$

Ta is a Σ_1^0 predicate and hence Ta can be viewed as a recursive enumeration of its numbers (cf. Lemma 2.1 of Simpson [14]). Clearly, $\bigcup\{Ta : a \in A\}$ covers C. So by compactness, it has a finite subcover $\{B_{ij} : i \leqslant k,\ j \leqslant l\}$ such that $B_{ij} \in Ta_i$, where a_i is the $(i+1)$-th element of A. From this subcover, we will construct a continuous function g on a finite dimensional set, which has a fixed point by Brouwer's theorem. Then from this fixed point, we will make $x \in C$ such that $\|f(x) - x\|_n \leqslant \epsilon$.

Suppose that $B_{ij} = (b_{ij}, r_{ij})$ and $\dim(b_{ij}) = m_{ij}$ for $i \leqslant k$ and $j \leqslant l$. Let $m = \max[\{m_{ij} : i \leqslant k \text{ and } j \leqslant l\} \cup \{n\}]$. For $x \in \mathbb{R}^{\mathbb{N}}$, $x[m]$ denotes the initial segment of x with length m. Let D be the convex hull of $\{a_i[m] : i \leqslant k\}$. We will construct a continuous function g on D.

For each $i \leqslant k$, we first define a continuous function $d_i : D \to \mathbb{R}$ by

$$d_i(x) = \max[\{r_{ij} - \|x - b_{ij}\|_{m_{ij}} : j \leqslant l\} \cup \{0\}].$$

Note that $d_i(x) > 0$ iff any extension of x to C belongs to B_{ij} for some $j \leqslant l$. It is not difficult (but somewhat messy) to construct a code for the continuous function d_i. So we leave this construction to the reader. It is also easy to see that $\sum_{j \leqslant k} d_j(x) > 0$ for each $x \in D$. We set

$$\beta_i(x) = \frac{d_i(x)}{\sum_{j \leqslant k} d_j(x)}.$$

Then for all $x \in D$, we have

$$\sum_{i \leqslant k} \beta_i(x) = 1.$$

We finally define a continuous function $g : D \to D$ by

$$g(x) = \sum_{i \leqslant k} \beta_i(x) a_i[m].$$

We now apply Brouwer's theorem II to the function $g : D \to D$. Let x be any fixed point of g, and $\bar{x}$ be a point in C such that $\bar{x}[m] = x$. Let $I = \{i \leqslant n : \beta_i(x) > 0\}$. Such an I exists by bounded Σ_1^0 separation (cf. Lemma 1.6 of [8]). We here notice that

$$\|f(\bar{x}) - a_i\|_n \leqslant \epsilon \quad \text{for all } i \in I,$$

since if $\beta_i(x) > 0$, $\bar{x}$ belongs to B_{ij} for some j and hence $\bar{x}$ belongs to Ta_i.

Since $\sum_{i\in I}\beta_i(x)=1$, we have

$$\begin{aligned}\|f(\bar{x})-\bar{x}\|_n &= \left\|\sum_{i\in I}\beta_i(x)f(\bar{x})-\sum_{i\in I}\beta_i(x)a_i\right\|_n\\ &\leq \sum_{i\in I}\beta_i(x)\,\|f(\bar{x})-a_i\|_n\leq\epsilon.\end{aligned}$$

This completes the proof. $\square$

Remark. The above theorem is indeed provable within RCA_0. For if the compact set C includes a line segment, the line segment is also compact, which implies WKL_0 by Lemma 3.1, and otherwise the theorem is trivial since C consists of a single point.

As an application of the above fixed point theorem, we prove, within WKL_0, the Cauchy–Peano theorem for ordinary differential equations. In [14], Simpson has already proved the Cauchy–Peano theorem in WKL_0 by eliminating the use of the Ascoli lemma from Peano's original proof. A key idea in his proof is that a solution of the initial value problem can be found in a set of equicontinuous (or Lipchitzian) functions, which can be encoded by points in $\mathbb{R}^{\mathbb{N}}$ (see Lemma 4.3 in this paper). In order to use the fixed point theorem, we need a precise description of the set in $\mathbb{R}^{\mathbb{N}}$ corresponding to the set of equicontinuous functions, while Simpson only uses the fact that the equicontinuous functions can be embedded into the compact set $[-1,1]^{\mathbb{N}}$. We here emphasize that our aim is not merely to prove the Cauchy–Peano theorem but also to develop a part of functional analysis related to fixed point theorems within second-order arithmetic.

6.2. Cauchy–Peano Theorem (WKL_0). *Let $f(x,y)$ be a continuous function from the rectangle $D=\{(x,y)\in\mathbb{R}^2: |x|\leq a,\ |y|\leq b\}$ to $\mathbb{R}$. Then the initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y),\qquad y(0)=0$$

has at least one solution $y=\phi(x)$ on the interval $[-\alpha,\alpha]$, where $\alpha=\min(a,b/M)$ and $M=\max\{|f(x,y)|:(x,y)\in D\}$. (*Note.* It is provable in WKL_0 that f has a maximum on D, see [13].)

Proof. Suppose that $\{q_i\}_{i\in\mathbb{N}}$ enumerates the rationals in $[-\alpha,\alpha]$ so that $q_0=0$ and $q_i\neq q_j$ $(i\neq j)$. We define a closed set C in $\mathbb{R}^{\mathbb{N}}$ by

$$\langle u_i\rangle\in C \quad\Leftrightarrow\quad u_0=0 \quad\text{and}\quad |u_i-u_j|\leq M\,|q_i-q_j| \quad\text{for all } i,j.$$

By Lemma 4.3, we may think that $u=\langle u_i\rangle\in C$ encodes a continuous function $\tilde{u}$: $[-\alpha,\alpha]\to\mathbb{R}$ such that $\tilde{u}(q_i)=u_i$ for all i. By identifying u with $\tilde{u}$, C can be regarded as the set of continuous functions $g:[-\alpha,\alpha]\to\mathbb{R}$ such that $g(0)=0$ and $|g(x)-g(y)|\leq M\,|x-y|$ for all $x,y\in[-\alpha,\alpha]$. It is easy to see in WKL_0 that C is compact and convex. To apply Theorem 6.1 to this C, we also need to show that C is countably represented by some $A \subseteq \mathbb{Q}^{\mathbb{N}}$.

Let $P = \{\langle p_0, \ldots, p_n\rangle \in \mathbb{Q}^{<\mathbb{N}} : p_0 = 0$ and $|p_i - p_j| < M\,|q_i - q_j|$ if $i \neq j\}$. For $p = \langle p_0, p_1, \ldots, p_n\rangle \in P$, we define a standard extension $\bar{p} = \langle \bar{p}_0, \bar{p}_1, \ldots, \bar{p}_n, \bar{p}_{n+1}, \ldots\rangle$ of p into $\mathbb{Q}^{\mathbb{N}}$ as follows: for each $k \geq 0$,

$$\bar{p}_k = \begin{cases} p_i & \text{if } q_k \in [q_i, \alpha],\ i \leq n \text{ and } \neg\exists l \leq n\ (q_l \in (q_i, \alpha]), \\ \dfrac{p_j - p_i}{q_j - q_i}(q_k - q_i) + p_i & \text{if } q_k \in [q_i, q_j),\ i, j \leq n \text{ and } \neg\exists l \leq n\ (q_l \in (q_i, q_j)), \\ p_j & \text{if } q_k \in [-\alpha, q_j),\ j \leq n \text{ and } \neg\exists l \leq n\ (q_l \in [-\alpha, q_j)). \end{cases}$$

We put $A = \{\bar{p} \in \mathbb{Q}^{\mathbb{N}} : p \in P\}$. Then A can be seen as the set of piecewise linear functions on $[-\alpha, \alpha]$. We will show that C is countably represented by A. Choose any $u = \langle u_i\rangle \in C$. We want to find a sequence $\langle \bar{p}^k : k \in \mathbb{N}\rangle$ from A such that $\|u - \bar{p}^k\|_{k+1} \leq 2^{-k}$ for each k. Fix $k \in \mathbb{N}$. We construct a sequence $p^k = \langle p_0, p_1, \ldots, p_k\rangle$ in P such that $|u_i - p_i| \leq 2^{-k}$ for each $i \leq k$, and then extend it to $\bar{p}^k \in A$. Let $n \in \mathbb{N}$ be large enough (strictly, $n \geq 2k + 2$ and $2^{k+1}/2^n \leq M\,|q_i - q_j|$ for all $i, j \leq k$ $(i \neq j)$). For each $i \leq k$, let r_i be a rational number such that $|u_i - r_i| < 2^{-n}$. Suppose $|r_{i_0}| \leq |r_{i_1}| \leq \cdots \leq |r_{i_k}|$ with $\{i_0, i_1, \ldots, i_k\} = \{0, 1, \ldots, k\}$. We now define $\langle p_0, p_1, \ldots, p_k\rangle$ as follows: for $j \leq k$,

$$p_{i_j} = \begin{cases} r_{i_j} - \dfrac{2^{j+1}}{2^n} & \text{if } r_{i_j} \geq \dfrac{2^{j+1}}{2^n}, \\ r_{i_j} + \dfrac{2^{j+1}}{2^n} & \text{if } r_{i_j} \leq -\dfrac{2^{j+1}}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that $|u_i - p_i| \leq 2^{-k}$ for all $i \leq k$. To show $\langle p_0, \ldots, p_k\rangle \in P$, we compute: for $j < l \leq k$,

$$\begin{aligned} |p_{i_j} - p_{i_l}| &\leq \max\left\{|r_{i_j} - r_{i_l}| - \frac{2^{l+1} - 2^{j+1}}{2^n}, \frac{2^{l+1} - 2^{j+1}}{2^n}\right\} \\ &< \max\left\{|u_{i_j} - u_{i_l}| + \frac{1}{2^n} + \frac{1}{2^n} - \frac{2^2 - 2^1}{2^n}, \frac{2^{k+1}}{2^n}\right\} \\ &\leq M\,|q_{i_j} - q_{i_l}|. \end{aligned}$$

It is also obvious that $p_0 = 0$. Hence $\langle p_0, \ldots, p_k\rangle$ is in P, and so it can be extended to $\bar{p}^k$ in A.

We next define a continuous function $F : C \to C$ as follows: for $u \in C$,

$$F(u) = \left\langle \int_0^{q_i} f(t, \bar{u}(t))\,\mathrm{d}t : i \in \mathbb{N}\right\rangle,$$

where $\bar{u}$ is a continuous function encoded by u. It is then obvious that $F(u) \in C$

for $u \in C$, since $\widetilde{F(u)}(0) = 0$ and for all x, y in $[-\alpha, \alpha]$,

$$|\widetilde{F(u)}(x) - \widetilde{F(u)}(y)| = \left|\int_x^y f(t, \tilde{u}(t))\, \mathrm{d}t\right| \leq M\,|x - y|.$$

Formally, a code Φ for F is given by

$(a, r, b, s) \in \Phi \quad \Leftrightarrow \quad a, b \in P$ and $r, s \in Q^+$, and if $n = \dim(a)$ and $-\alpha \leq q_{i_0} < q_{i_1} \cdots < q_{i_{n-1}} \leq \alpha$ with $\{i_0, \ldots, i_{n-1}\} = \{0, \ldots, n-1\}$,

then there exists $e \in \mathbb{N}$ such that

(i) $\alpha - q_{i_{n-1}}$, $q_{i_j} - q_{i_{j-1}}$ $(1 \leq j \leq n-1)$, $q_{i_0} + \alpha$ are all $< \dfrac{1}{M \cdot 2^{h(e)+2}}$,

(ii) $r < 2^{-h(e)-2}$,

(iii) $\|F(\bar{a}) - b\|_{\dim(b)} < s - \alpha \cdot 2^{-e}$,

where h is a modulus of uniform continuity for f. Conditions (i) and (ii) together imply that for any point $u = \langle u_i \rangle \in B_r(a) \cap C$, $|u_i - \bar{a}_i| < 2^{-h(e)}$ for all i, where $\bar{a} = \langle \bar{a}_i \rangle$ is the standard extension of a. Then $|f(t, \tilde{u}(t)) - f(t, \tilde{\bar{a}}(t))| \leq 2^{-e}$ for all $t \in [-\alpha, \alpha]$, and so $|\widetilde{F(u)}(x) - \widetilde{F(\bar{a})}(x)| \leq |x| \cdot 2^{-e} \leq \alpha \cdot 2^{-e}$. Therefore, $F(u) \in B_s(b)$ by (iii), which means that Φ is a code for F. We leave the details to the reader. At last, the fixed point of F clearly gives a solution of the initial value problem. □

7. Markov–Kakutani fixed point theorem and Hahn–Banach theorem

The Markov–Kakutani fixed point theorem asserts the existence of a common fixed point for certain families of affine mappings. A continuous function T from a convex closed set C to itself is said to be *affine* if $T(qx + (1-q)y) = qT(x) + (1-q)T(y)$ whenever $q \in [0, 1] \cap \mathbb{Q}$ and x, $y \in C$. This theorem has numerous applications [4], [17]. Among others, Kakutani [11] has proved the Hahn–Banach theorem from this type of fixed point theorem (see also [10], [18]). In this section, we prove the Markov–Kakutani theorem for $\mathbb{R}^{\mathbb{N}}$ within RCA_0, and use it to prove the Hahn–Banach theorem for separable Banach spaces within WKL_0.

7.1. Markov–Kakutani Theorem (RCA_0). *Let C be a nonempty compact convex closed set in $\mathbb{R}^{\mathbb{N}}$. Let $\langle T_n \rangle$ be a sequence of continuous functions from C to C such that*

(i) *for each n, T_n is affine on C,*

(ii) *for each m and n, $T_n \circ T_m(x) = T_m \circ T_n(x)$ $(x \in C)$.*

Then there exists $x \in C$ such that $T_n(x) = x$ for all n.

Proof. Let C and $\langle T_n \rangle$ be as in the above statement. For each $n \in \mathbb{N}$, let $K_n = \{x \in \mathbb{R}^{\mathbb{N}} : T_n x = x\} \cap C$. Formally, we express the complement of K_n in $\mathbb{R}^{\mathbb{N}}$ as the union of the open set $\{B : (B, B') \in \Phi_n$ and $B \cap B' = \emptyset\}$ and the complement of C, where Φ_n is a code for T_n. Then K_n is a closed set in $\mathbb{R}^{\mathbb{N}}$. Our goal is to show that $\cap \{K_n : n \in \mathbb{N}\}$ is nonempty.

Let $K_{n,i} = \{x \in \mathbb{R}^{\mathbb{N}} : \|T_n x - x\|_{i+1} \leq 2^{-i}\} \cap C$. We can easily see that $K_{n,i}$ is well defined as a closed set in $\mathbb{R}^{\mathbb{N}}$. Since $K_n = \bigcap \{K_{n,i} : i \in \mathbb{N}\}$ for each n, we want to show that $\bigcap \{K_{n,i} : n \in \mathbb{N},\ i \in \mathbb{N}\} \neq \emptyset$.

By way of contradiction, we assume $\bigcap \{K_{n,i} : n \in \mathbb{N},\ i \in \mathbb{N}\} = \emptyset$. Then $\bigcup \{$the complement of $K_{n,i}$ in $\mathbb{R}^{\mathbb{N}} : n \in \mathbb{N},\ i \in \mathbb{N}\} = \mathbb{R}^{\mathbb{N}} \supset C$. By the compactness of C, there exist $k \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $\bigcup \{$the complement of $K_{n,i}$ in $\mathbb{R}^{\mathbb{N}} : n \leq k$, $i \leq l\} \supset C$. Hence $\bigcap \{K_{n,i} : n \leq k,\ i \leq l\} \cap C = \emptyset$, and so $\bigcap \{K_{n,i} : n \leq k,\ i \leq l\} = \emptyset$, since each $K_{n,i} \subseteq C$.

Let z be any element of C. Choose $p \in \mathbb{N}$ such that $\|x - y\|_{l+1} \leq p \cdot 2^{-l}$ for all x, $y \in C$. The existence of such a p is clear from the compactness of C. We define a point x in C by

$$x = \frac{1}{p^{k+1}} \sum_{i_0=0}^{p-1} \cdots \sum_{i_k=0}^{p-1} T_0^{i_0} \cdots T_k^{i_k} z,$$

where $T_n^{i+1}(z) = T_n(T_n^i(z))$ and $T_n^0(z) = z$. We will show $x \in \bigcap \{K_{n,i} : n \leq k,\ i \leq l\}$ for a contradiction. Fix any $n \leq k$. Putting

$$x_n = \frac{1}{p^k} \sum_{i_0=0}^{p-1} \cdots \sum_{i_{n-1}=0}^{p-1} \sum_{i_{n+1}=0}^{p-1} \cdots \sum_{i_k=0}^{p-1} T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} T_{n+1}^{i_{n+1}} \cdots T_k^{i_k} z,$$

we have

$$\begin{aligned} \|T_n x - x\|_{l+1} &= \left\| T_n\left(\frac{1}{p} \sum_{i_n=0}^{p-1} T_n^{i_n} x_n\right) - \frac{1}{p} \sum_{i_n=0}^{p-1} T_n^{i_n} x_n \right\|_{l+1} \\ &= \left\| \frac{1}{p} \sum_{i_n=0}^{p-1} T_n^{i_n+1} x_n - \frac{1}{p} \sum_{i_n=0}^{p-1} T_n^{i_n} x_n \right\|_{l+1} \\ &= \frac{1}{p} \|T_n^p x_n - x_n\|_{l+1} \leq \frac{1}{p}(p \cdot 2^{-l}) = 2^{-l}. \end{aligned}$$

Remark that we can easily show by Π_1^0 induction (cf. the proof of Lemma 3.2) that if $T : C \to C$ is affine, then for any finite subset $\{x_0, \ldots, x_{n-1}\} \subseteq C$, and for any set of non-negative reals $\{\alpha_0, \ldots, \alpha_{n-1}\}$ with $\sum_{i<n} \alpha_i = 1$, we have $f(\sum_{i<n} \alpha_i x_i) = \sum \alpha_i f(x_i)$. From the above inequality, $x \in K_{n,l} = \bigcap \{K_{n,i} : i \leq l\}$. Since n is any number $\leq k$, $x \in \bigcap \{K_{n,i} : n \leq k,\ i \leq l\}$. This is a contradiction. So we are done. $\square$

To state the Hahn–Banach theorem, we need introduce some basic notions on Banach spaces. We define a code for a *separable Banach space* to be a nonempty set $A \subseteq \mathbb{N}$ together with operations $+$, $- : A \times A \to A$, $\cdot : Q \times A \to A$, $\|\cdot\| : A \to [0, \infty)$ and a distinguished element $0 \in A$ such that A, $+$, $-$, 0 forms a vector space over $\mathbb{Q}$ and $\|\cdot\|$ satisfies $\|qa\| = |q|\,\|a\|$ and $\|a+b\| \leq \|a\| + \|b\|$ for all a, $b \in A$, $q \in \mathbb{Q}$. A *point of the separable Banach space* $\hat{A}$ is defined to be a sequence $\langle a_n : n \in \mathbb{N}\rangle$ from A, satisfying $\forall n\, \forall i\, (\|a_n - a_{n+i}\| \leq 2^{-n})$. Let $\hat{A}$ and $\hat{B}$ be a separable Banach spaces. A *continuous function* $F:\hat{A} \to \hat{B}$ is encoded as a sequence of $(a, r, b, s) \in A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ satisfying the three conditions analogous to those for a continuous function on $\mathbb{R}^n$. A *bounded linear operator* is a continuous function $F:\hat{A} \to \hat{B}$ such that

(i) $F(pa + qb) = pF(a) + qF(b)$ for all a, $b \in A$ and p, $q \in \mathbb{Q}$,

(ii) there exists a real number α such that $\|F(a)\| \leq \alpha\,\|a\|$ for all $a \in A$.

If $\alpha \in \mathbb{R}$ satisfies $\|F(a)\| \leq \alpha\,\|a\|$ for all $a \in A$, we write $\|F\| \leq \alpha$. A bounded linear operator $F:\hat{A} \to \hat{B}$ is also called a *bounded linear functional* if $\hat{B} = \mathbb{R}$. A bounded linear operator $F:\hat{A} \to \hat{B}$ can be encoded by the restriction of F to A (see Lemma 5.4 in Brown and Simpson [3] and Lemma 4.3 in this paper). If there is a bounded linear operator Ψ from $\hat{A}$ to $\hat{B}$ satisfying $\|\Psi(a)\| = \|a\|$ for all $a \in A$, then $\hat{A}$ is called a *closed linear subspace* of $\hat{B}$, and a point x in $\hat{A}$ is identified with $\Psi(x)$ in $\hat{B}$.

We now state the Hahn–Banach theorem for separable Banach spaces, and prove it within WKL_0.

7.2. Hahn–Banach Theorem (WKL_0). *Let $\hat{S}$ be a closed linear subspace of the separable Banach space $\hat{A}$. Let $F:\hat{S} \to \mathbb{R}$ be a bounded linear functional such that $\|F\| \leq 1$. Then there exists a bounded linear functional $\tilde{F}:\hat{A} \to \mathbb{R}$ extending F and such that $\|\tilde{F}\| \leq 1$.*

Proof. Let $\hat{S}$ and $\hat{A}$ be separable Banach spaces. Let $\Psi:\hat{S} \to \hat{A}$ be a bounded linear operator satisfying $\|\Psi(x)\| = \|x\|$ for all $x \in S$. Let $F:\hat{S} \to \mathbb{R}$ be a bounded linear functional such that $\|F\| \leq 1$. We want to obtain a bounded linear functional $\tilde{F}:\hat{A} \to \mathbb{R}$ such that $F(x) = \tilde{F}(\Psi(x))$ for all $x \in \hat{S}$, and such that $\|\tilde{F}\| \leq 1$.

Suppose that $A = \{a_i\}$, $S = \{s_i\}$, $a_0 = 0$ and $s_0 = 0$. We define a closed set C_0 in $\mathbb{R}^{\mathbb{N}}$ by

$$\langle x_i\rangle \in C_0 \quad \Leftrightarrow \quad x_0 = 0 \text{ and } |x_i - x_j| \leq \|a_i - a_j\| \quad \text{for all } i, j.$$

We can easily see in WKL_0 that C_0 is compact and convex. Let $P = \{\langle u_0, \ldots, u_n\rangle \in \mathbb{Q}^{<\mathbb{N}} : u_0 = 0 \text{ and } (|u_i - u_j| < \|a_i - a_j\| \text{ or } u_i = u_j)\}$ and $X = \{\langle \bar{u}_i\rangle \in \mathbb{R}^{\mathbb{N}} : \langle u_0, \ldots, u_n\rangle \in P \text{ and } \bar{u}_i = \min\{u_j + \|a_i - a_j\| : j \leq n\}\}$. As in the proof of Theorem 6.2, we can easily show that C_0 is countably represented by X. By (a generalization of) Lemma 4.3, we can identify $g = \langle g_i\rangle \in C_0$ with the continuous function $\tilde{g}:\hat{A} \to \mathbb{R}$ such that $\tilde{g}(a_i) = g_i$. So C_0 may be regarded as the set of continuous functions $\tilde{g}:\hat{A} \to \mathbb{R}$ such that $\tilde{g}(a_0) = 0$ and $|\tilde{g}(x) - \tilde{g}(y)| \leq \|x - y\|$ for all x, $y \in \hat{A}$. A desired functional $\tilde{F}$ will be found in C_0.

Our proof goes as follows. We define three closed subsets of C_0, $C_1 \supset C_2 \supset C_3$. Roughly speaking, C_1 is the set of continuous functions $\tilde{f}$ in C_0 extending F, C_2

the set of continuous functions $\bar{f}$ in C_1 such that $\bar{f}(x+y)=\bar{f}(x)+\bar{f}(y)$ for all $x \in \hat{A}$ and $y \in \hat{S}$, C_3 the set of continuous functions $\bar{f}$ in C_2 such that $\bar{f}(x+y)=\bar{f}(x)+\bar{f}(y)$ for all $x \in \hat{A}$ and $y \in \hat{A}$, and such that $\bar{f}(\alpha x)=\alpha\bar{f}(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \hat{A}$. Clearly, any function in C_3 is a desired functional $\bar{F}$. So what we need is to show that C_3 is nonempty. To show the nonemptiness of C_2 and C_3, we will apply Theorem 7.1 to certain families of continuous functions on C_1 and C_2, respectively.

First let $C_1=\{f \in C_0 : \bar{f}(\Psi(s_i))=F(s_i) \text{ for all } i\}$. Formally, we should express the complement of C_1 as an infinite sequence of (codes for) basic open sets. We however omit this routine work. To prove that C_1 is not empty, we set $C_{1,n}=\{f \in C_0 : \bar{f}(\Psi(s_i))=F(s_i) \text{ for all } i \leqslant n\}$ and show $C_{1,n} \neq \emptyset$ for each n. Choose any n. We define a point $\langle x_k \rangle \in \mathbb{R}^{\mathbb{N}}$ by

$$x_k=\min\{F(s_i)+\|a_k-\Psi(s_i)\| : i \leqslant n\}.$$

Then we have

$$x_k \leqslant F(s_0)+\|a_k-\Psi(s_0)\|=\|a_k\|,$$
$$x_k \geqslant \min\{-\|\Psi(s_i)\|+\|a_k-\Psi(s_i)\| : i \leqslant n\} \geqslant -\|a_k\|.$$

In particular, $x_0=\|a_0\|=0$. We also have

$$\begin{aligned} x_k-x_l &\leqslant \min\{F(s_i)+\|a_k-a_l\|+\|a_l-\Psi(s_i)\| : i \leqslant n\}-x_l \\ &=\|a_k-a_l\|, \\ x_k-x_l &\geqslant x_k-\min\{F(s_i)+\|a_l-a_k\|+\|a_k-\Psi(s_i)\| : i \leqslant n\} \\ &=-\|a_k-a_l\|. \end{aligned}$$

Thus $\langle x_k \rangle \in C_0$. Let $\bar{g} : \hat{A} \to \mathbb{R}$ be the continuous function such that $\bar{g}(a_k)=x_k$. Then for each $j \leqslant n$,

$$\bar{g}(\Psi(s_j)) \leqslant F(s_j)+\|\Psi(s_j)-\Psi(s_j)\|=F(s_j),$$
$$\bar{g}(\Psi(s_j)) \geqslant \min\{F(s_i)+F(s_j-s_i) : i \leqslant n\}=F(s_j).$$

So $g \in C_{1,n}$. Since C_0 is compact and $C_1=\bigcap_n C_{1,n}$, it follows that $C_1 \neq \emptyset$. Moreover, it is obvious that C_1 is compact and convex.

Next let $C_2=\{f \in C_1 : \bar{f}(a_i+\Psi(s_j))=\bar{f}(a_i)+\bar{f}(\Psi(s_j)) \text{ for all } i, j\}$. We want to show $C_2 \neq \emptyset$. Define a family of continuous functions $\{T_j : C_1 \to C_1\}$ by

$$(T_j f)(a_i)=\bar{f}(a_i+\Psi(s_j))-\bar{f}(\Psi(s_j)).$$

Formally, a code Φ for $\langle T_j \rangle$ is given by

$(j, u, p, v, q) \in \Phi \iff j \in \mathbb{N}, u, v \in P$ and $p, q \in \mathbb{Q}^+$, and

if $n=\dim(u)$, $m=\dim(v)$ and $\Psi(s_j)=\langle a_{j_k} : k \in \mathbb{N} \rangle$ then

(i) $\forall i<m\ \exists l<n\ (a_l=a_i+a_{j_{m-1}})$,

(ii) $p<2^{-m+1}$,

(iii) $\|T_j\bar{u}-v\|_m<q-6 \cdot 2^{-m+1}$,

where $\bar{u} = \langle \bar{u}_i \rangle$ is defined by $\bar{u}_i = \min\{u_j + \|a_i - a_j\| : j < n\}$. We omit checking that Φ really encodes $\langle T_j \rangle$. We show that the range of T_j is included in C_1 as follows: for each $f \in C_1$, $T_j f \in \hat{X}$, since

$$|(T_j f)(a_i) - (T_j f)(a_k)| = |\tilde{f}(a_i + \Psi(s_j)) - \tilde{f}(a_k + \Psi(s_j))| \leq \|a_i - a_k\|,$$

and for each $f \in C_1$, $T_j f \in C_1$, since

$$\begin{aligned}(T_j f)(\Psi(s_k)) &= \tilde{f}(\Psi(s_k) + \Psi(s_j)) - \tilde{f}(\Psi(s_j)) \\ &= F(s_k + s_j) - F(s_j) = F(s_k).\end{aligned}$$

It is obvious that each T_j is affine and $T_j \circ T_k = T_k \circ T_j$. So by Theorem 7.1, there exists $g \in C_1$ such that $\tilde{g}(a_i) = \tilde{g}(a_i + \Psi(s_j)) - \tilde{g}(\Psi(s_j))$ for all i, j. Then $C_2 \neq \emptyset$. Moreover, C_2 is compact and convex.

Finally, let $C_3 = \{f \in C_2 : \tilde{f}(a_i + a_j) = \tilde{f}(a_i) + \tilde{f}(a_j) \text{ for all } i, j\}$. Define a family of continuous functions $\{U_j : C_2 \to C_2\}$ by

$$(U_j f)(a_i) = \tilde{f}(a_i + a_j) - \tilde{f}(a_j).$$

A code for $\langle U_j \rangle$ can be encoded in the same way as for $\langle T_j \rangle$. It is easy to see that for each $f \in C_2$, $U_j f \in C_0$. To see $U_j f \in C_1$, we have

$$(U_j f)(\Psi(s_i)) = \tilde{f}(\Psi(s_i) + a_j) - \tilde{f}(a_j) = \tilde{f}(\Psi(s_i)) = F(s_i).$$

We leave a check for $U_i f \in C_2$ to the reader. It is also easy to see that each U_j is affine and $U_j \circ U_k = U_k \circ U_j$. So by Theorem 7.1, there exists $g \in C_3$ such that $\tilde{g}(a_i) = \tilde{g}(a_i + a_j) - \tilde{g}(a_j)$ for all i, j. Thus $C_3 \neq \emptyset$.

By Π^0_1-induction (equivalently Σ^0_1-induction), we can easily show in RCA_0 that if $f \in C_3$ then

$$\tilde{f}(na_i) = n\tilde{f}(a_i) \quad \text{for all } n \in \mathbb{N}.$$

Hence, if $f \in C_3$ then

$$m\tilde{f}\left(\frac{n}{m} a_i\right) = \tilde{f}(na_i) = n\tilde{f}(a_i) \quad \text{for } n, m \in \mathbb{N},$$

that is,

$$\tilde{f}(qa_i) = q\tilde{f}(a_i) \quad \text{for } q \in \mathbb{Q}.$$

Therefore, any continuous function in C_3 is a bounded linear functional extending F. This completes the proof. □

For the reversal of the Hahn–Banach theorem, see Metakides, Nerode and Shore [12], and Brown and Simpson [3]. They have indeed proved that the Hahn–Banach theorem and WKL_0 are equivalent to each other over RCA_0.

Note. Stephen Simpson and the second author have recently shown that in Theorem 6.1, the assumption $A \subseteq \mathbb{Q}^{\mathbb{N}}$ can be weakened to $A \subseteq \mathbb{R}^{\mathbb{N}}$, which makes the proof of Theorem 6.2 a little easier. Lemma 4.3, Theorems 5.3 and 5.4 can be also generalized similarly. These results will appear in a future publication.

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